

Addendum: Observables for General Relativity related to geometry

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ABSTRACT: In this addendum we clarify a point which strengthens one of the results from [1]. Namely, we show that the algebra of the observables $F(r, \theta)$ is yet simpler than it was described in [1]. This is an important point, because with this simplification an important subalgebra becomes canonical, allowing for a natural reduction of the phase space.

1 Introduction

In equations (4.11) from [1] we computed the Poisson brackets between observables not constructed from the momentum conjugate to the metric (i.e., $\Phi_\alpha, \Pi^\alpha, Q_{AB}$) with the observables corresponding to that momentum (i.e., P^{AB}).¹ In each case the result contained terms proportional to a delta at zero. In fact, this result can be strengthened a bit, by realizing that in an important class of cases the terms proportional to the delta at zero on the right-hand sides of (4.11) do not contribute and therefore are zero. This is an important simplification of the algebra, because without those delta terms and knowing additionally that the Poisson bracket $\{P^{AB}, P^{CD}\}$ is vanishing the algebra of the observables $\Phi_\alpha, \Pi^\alpha, Q_{AB}, P^{AB}$ is truly canonical.

In this addendum, we will first analyze the expressions in (4.11) and show that the vanishing of those terms can be seen already at this level. In fact, a closer look at (4.11a) will be enough to provide an argument for the rest of the equations. Next, we will compute the Poisson bracket $\{P^{AB}, P^{CD}\}$.

¹Note that we are adopting the notation from [1] throughout this addendum. This means in particular, that the indices A, B, \dots correspond to the angular components of the appropriate tensors.

2 Vanishing of the $\delta(r')$ terms

Let us recall the result spelled out in (4.11a)

$$\{\Phi_\alpha(r, \theta), P^{AB}(r', \theta')\} = -\frac{1}{2}y'^A_I y'^B_J T^{IJ}_{KL} h^{LM} x^K y^C_M \partial_C \Phi_\alpha(r, \theta) \delta(r'). \quad (2.1)$$

Notice, that this equation, when smeared with a smooth tensor field w_{IJ} , such that

$$w_{ra} = 0 \quad (2.2)$$

gives

$$\int dr' d^2\theta' w_{AB}(r', \theta') \{\Phi_\alpha(r, \theta), P^{AB}(r', \theta')\} = -\frac{1}{2}w_{IJ}(0) T^{IJ}_{KL} h^{LM} x^K y^C_M \partial_C \Phi_\alpha(r, \theta). \quad (2.3)$$

However, for fields w_{IJ} satisfying condition (2.2) we have

$$w_{IJ}(0) = \lim_{r \rightarrow 0} \left(n_I(\theta) n_J(\theta) w_{rr}(r, \theta) \right) = 0, \quad (2.4)$$

and therefore the right hand side of (2.3) vanishes. Since the components w_{AB} were arbitrary we conclude that in fact

$$\{\Phi_\alpha(r, \theta), P^{AB}(r', \theta')\} = 0. \quad (2.5)$$

This fact can also be phrased in the following way. The left hand side of (2.1) is smeared in primed variables with tensor fields vanishing at zero (since for any smooth tensor field w_{IJ} we have $w_{AB} \sim r^2$, for a smooth tensor field satisfying additionally equation (2.2) we have $w_{AB} \sim r^3$) and therefore the delta at zero does not contribute to the result. One might worry that when we consider observables which contain radial derivatives of the momentum, namely $\partial_r^{(n)} P^{AB}$, we will see some contributions. It turns out, that for smooth tensor fields w_{IJ} satisfying (2.2) which are symmetric (namely $w_{IJ} = w_{JI}$, which is the case here since P^{IJ} is symmetric) we have $w_{AB} \sim r^4$, therefore even for observables containing the derivative $\partial_r P^{AB}$ the delta term does not contribute in their Poisson brackets.

The behaviour at zero of the w_{IJ} used above can be justified in the following way. For a smooth tensor field w_{IJ} satisfying (2.2), we can infer

$$w_{IJ}(0) = 0. \quad (2.6)$$

This implies

$$w_{IJ} = x^K f_{KIJ}, \quad (2.7)$$

for some smooth tensor field f_{KIJ} . On one hand, from (2.2) we get

$$x^K x^I f_{KIJ} = 0, \quad (2.8)$$

and therefore, taking the second derivative at zero, we get

$$f_{(KI)J}(0) = 0, \quad (2.9)$$

which means

$$f_{KIJ}(0) = f_{[KI]J}(0). \quad (2.10)$$

On the other hand,

$$f_{KIJ}(0) = \partial_K w_{IJ}(0) \quad (2.11)$$

and hence, invoking the fact that w_{IJ} is symmetric,

$$f_{KIJ}(0) = f_{K(IJ)}(0). \quad (2.12)$$

Combining (2.10) and (2.12) we get that

$$f_{KIJ}(0) = 0, \quad (2.13)$$

which means

$$\partial_K w_{IJ}(0) = 0 \quad (2.14)$$

justifying the statement that for the tensor fields w_{IJ} of interest here, we have $w_{AB} \sim r^4$.

It is straightforward to apply the above arguments in all of the equations from (4.11) leaving them in the form

$$\{\Phi_\alpha(r, \theta), P^{AB}(r', \theta')\} = 0, \quad (2.15a)$$

$$\{\Pi^\alpha(r, \theta), P^{AB}(r', \theta')\} = 0, \quad (2.15b)$$

$$\{Q_{CD}(r, \theta), P^{AB}(r', \theta')\} = \delta_{(C}^A \delta_{D)}^B \delta(r - r') \delta(\theta - \theta'). \quad (2.15c)$$

3 The Poisson bracket $\{P^{AB}, P^{CD}\}$

Let us compute the following Poisson bracket

$$\{P^{AB}(r', \theta'), \int d^3x w_{IJ}(x) P^{IJ}(x)\} = \int d^3\sigma \int d^3x \frac{\delta D_{P^{AB}(r', \theta')}}{\delta q_{ra}(\sigma)} \frac{\delta P^{IJ}(x)}{\delta p^{ra}(\sigma)} w_{IJ}(x) \quad (3.1)$$

$$= \int d^3\sigma \frac{\delta D_{P^{AB}(r', \theta')}}{\delta q_{ra}(\sigma)} w_{ra}(\sigma). \quad (3.2)$$

Choosing w_{IJ} to be such a tensor field that the w_{ra} components vanish, we obtain

$$\{P^{AB}(r', \theta'), P^{CD}(r, \theta)\} = 0. \quad (3.3)$$

Recalling equations (4.5) from [1], namely,

$$\{\Phi_\alpha(r, \theta), \Pi^{\alpha'}(r', \theta')\} = \delta_\alpha^{\alpha'} \delta(r - r') \delta(\theta - \theta'), \quad (3.4a)$$

$$\{\Phi_\alpha(r, \theta), \Phi_{\alpha'}(r', \theta')\} = \{\Pi^\alpha(r, \theta), \Pi^{\alpha'}(r', \theta')\} = \{Q_{AB}(r, \theta), Q_{CD}(r', \theta')\} = 0, \quad (3.4b)$$

$$\{Q_{AB}(r, \theta), \Phi_\alpha(r', \theta')\} = \{Q_{AB}(r, \theta), \Pi^\alpha(r', \theta')\} = 0, \quad (3.4c)$$

we conclude that the algebra of observables $\Phi_\alpha, \Pi^\alpha, Q_{AB}, P^{AB}$ is canonical.

4 Final remarks

To see the consequences of the simplification of the Poisson algebra presented above, we should address its applications. Note, that in the ADM formulation of General Relativity we are interested in, the vector constraint can be written in the form

$$C[\vec{N}] = 2 \int P^{IJ} \nabla_I N_J, \quad (4.1)$$

so the field smearing the momentum is in this case $2\nabla_{(I} N_{J)}$. Though in general the value of $2\nabla_{(I} N_{J)}$ at zero is dictated by the shift vector and may be nonvanishing, it turns out, that a gauge fixing of General Relativity related to the observables discussed here is possible in which the shift vector is fixed in such a way that exactly $2\nabla_{(I} N_{J)}(0) = 0$. Following the discussion presented above we see that the condition $w_{IJ}(0) = 0$ is enough to draw the conclusion that the right hand side of (2.1) does not contribute to the Poisson bracket of Φ_α with the vector constraint. Arguing in the same manner we can see that the delta term does not contribute in the Poisson brackets of the vector constraint with all the observables.

The strength of the observables constructed in [1] is that they have a very clear geometrical interpretation. On the other hand, they depend nonlocally on the canonical data and this might be worrisome when one thinks of their possible applications. However, a reduction of the phase space to observables $\Phi_\alpha, \Pi^\alpha, Q_{AB}, P^{AB}$ advocated in [1] and mentioned in the previous paragraph can be completed. As this addendum shows, the reduced variables can have canonical Poisson brackets, and the nonlocality will be reappearing only in the Hamiltonian. The detailed construction of the reduced phase space is presented in [2]. Moreover, the reduced phase space can be used in quantisation, which is discussed in [3] and [4].

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